Radiative Transport Analysis for Plane Geometry with Isotropic Scattering and Arbitrary Temperature

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Particular solutions to the radiative transport equation are presented for an absorbing, emitting, and isotropic scattering medium with an arbitrary, but specified, temperature profile. The radiative transport is assumed to be one-dimensional and axisymmetric. Derivation of the particular solutions is based upon the method of variation of parameters. Homogeneous and particular solutions are derived from the discrete ordinate form of the radiative transport equation. The integration constants associated with the particular solution are expressed explicitly. This work yields a general solution which is expressed in closed form and is applicable to a wide range of problems involving radiative transport in absorbing, emitting, and scattering media. Also an illustrative example is presented for a medium having a linear temperature profile.

Introduction

THE equation for one-dimensional, axisymmetric radiative transfer in an absorbing, emitting, and isotropically scattering medium has the form

$$\frac{dI(\tau,\mu)}{d\tau} = -\frac{I(\tau,\mu)}{\mu} + \frac{W}{2\mu}\int_{-1}^{1} I(\tau,\mu')\,d\mu' + \frac{(1-W)n^2}{\mu} I_b(\tau) \tag{1}$$

where W = albedo, τ = optical depth, $\mu = \cos\theta$, n = refractive index, $I_b(\tau)$ = Planck's blackbody intensity function, and I = radiative intensity (see Fig. 1). One standard method of solution has been the method of discrete ordinates whereby the integral term in Eq. (1) is approximated by a Gaussian quadrature.[1-3] This generates a system of simultaneous linear ordinary differential equations given by

$$\frac{dI(\tau,\mu_i)}{d\tau} = -\frac{I(\tau,\mu_i)}{\mu_i} + \frac{W}{2\mu_i}\sum_{j=1}^{m} I(\tau,\mu_j)a_j + \frac{(1-W)n^2 I_b(\tau)}{\mu_i}, \quad i = 1,\ldots,m \tag{2}$$

where μ_i are the quadrature points, a_j are the quadrature weights, and m (which is an even integer) is the order of quadrature. The left-hand side of Eq. (2) and the first two terms on the right-hand side of Eq. (2) give rise to homogeneous solutions, $I_h(\tau,\mu_i)[i = 1,\ldots,m]$, and the last term on the right-hand side of Eq. (2) requires particular solutions, $I_p(\tau,\mu_i)[i = 1,\ldots,m]$. Thus once the homogeneous and particular solutions are known the general solution is then given by

$$I(\tau,\mu_i) = I_h(\tau,\mu_i) + I_p(\tau,\mu_i), \quad i = 1,\ldots,m \tag{3}$$

The homogeneous solution of Eq. (2) is already known[4,5]; also, as is obvious, the homogeneous solution is equal to the general solution when $W = 1.0$. The purpose of this paper is to find the particular solutions of Eq. (2) for $W \neq 1.0$ and also when an arbitrary function $I_b(\tau)$ in Eq. (2) is specified. This is equivalent to having a specified temperature profile $T(\tau)$, since I_b is a function of temperature $I_b[T(\tau)]$. Since the homogeneous solutions of Eq. (2) are known, the particular solutions may be constructed from the homogeneous solutions by the method of variation of parameters.[6]

Analysis

The homogeneous solution of Eq. (2) for $W \neq 1.0$ as presented in Ref. 5 may be written as

$$I_h(\tau,\mu_i) = \sum_{j=1}^{m/2} (1-\lambda_j\mu_j)\left\{\frac{C_j e^{\lambda_j\tau}}{1+\lambda_j\mu_i} + \frac{C_{m+1-j}e^{-\lambda_j\tau}}{1-\lambda_j\mu_i}\right\}, \quad i = 1,\ldots,m \tag{4}$$

where C_j and C_{m+1-j} are the m integration constants and λ_j are the $(m/2)$ positive eigenvalues which may be obtained from any of the following equivalent expressions

$$\sum_{k=1}^{m/2} a_k\mu_k{}^2\lambda_j{}^2/(1-\mu_k{}^2\lambda_j{}^2) = 1/W - 1 \tag{5}$$

$$\sum_{k=1}^{m/2} a_k/(1-\lambda_j{}^2\mu_k{}^2) = 1/W \tag{6}$$

or

$$\sum_{k=1}^{m} a_k/(1-\lambda_j\mu_k) = 2/W \tag{7}$$

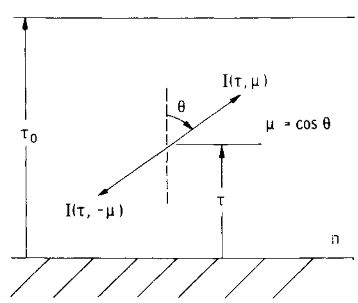

Fig. 1 Coordinate system and geometry.

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The particular solutions may be found by the method of variation of parameters,[6] which is based on knowledge of the homogeneous solutions. The particular solutions $I_p(\tau,\mu_i)[i = 1,\ldots,m]$ are given by

$$\begin{bmatrix} I_p(\tau,\mu_1) \\ \vdots \\ I_p(\tau,\mu_{m/2}) \\ I_p(\tau,-\mu_{m/2}) \\ \vdots \\ I_p(\tau,-\mu_1) \end{bmatrix} = \begin{bmatrix} x_{11}\cdots\cdots x_{1m} \\ \vdots \qquad \vdots \\ \vdots \qquad \vdots \\ \vdots \qquad \vdots \\ \vdots \qquad \vdots \\ x_{m1}\cdots\cdots x_{mm} \end{bmatrix} \begin{bmatrix} v_1(\tau) \\ \vdots \\ v_{m/2}(\tau) \\ v_{m/2+1}(\tau) \\ \vdots \\ v_m(\tau) \end{bmatrix} \tag{8}$$

where the x elements are known from the homogeneous solution, Eq. (4), and are given by

$$\left.\begin{aligned} x_{i,j} &= (1-\lambda_j\mu_j)C_j e^{\lambda_j\tau}/(1+\lambda_j\mu_i) \\ x_{i,m+1-j} &= (1-\lambda_j\mu_j)C_{m+1-j}e^{-\lambda_j\tau}/(1-\lambda_j\mu_i) \end{aligned}\right\} \begin{aligned} i &= 1,\ldots,m \\ j &= 1,\ldots,m/2 \end{aligned} \tag{9}$$

and the parameters, $v_j(\tau)[j = 1,\ldots,m]$, are to be determined from the system of differential equations

$$\begin{bmatrix} x_{11}\cdots\cdots x_{1m} \\ \vdots \qquad \vdots \\ \vdots \qquad \vdots \\ x_{m1}\cdots\cdots x_{mm} \end{bmatrix} \begin{bmatrix} dv_1/d\tau \\ \vdots \\ \vdots \\ dv_m/d\tau \end{bmatrix} = (1-W)n^2 I_b(\tau) \begin{bmatrix} 1/\mu_1 \\ \vdots \\ \vdots \\ 1/\mu_m \end{bmatrix} \tag{10}$$

where $\mu_k = -\mu_{m+1-k}[k = 1,\ldots,m/2]$. The solution of Eq. (10) for the element $(dv_{m+1-k})/d\tau[k = 1,\ldots,m/2]$ will have the form

$$\frac{dv_{m+1-k}}{d\tau} = \frac{\det\left[\dfrac{(1-\lambda_1\mu_1)C_1e^{\lambda_1\tau}}{1+\lambda_1\mu_j},\ldots,\dfrac{(1-\lambda_{m/2}\mu_{m/2})C_{m/2}e^{\lambda_{m/2}\tau}}{1+\lambda_{m/2}\mu_j},\ldots,\dfrac{1}{\mu_j},\ldots,\dfrac{(1-\lambda_1\mu_1)C_me^{-\lambda_1\tau}}{1-\lambda_1\mu_j}\right]}{\det\left[\dfrac{(1-\lambda_1\mu_1)C_1e^{\lambda_1\tau}}{1+\lambda_1\mu_j},\ldots,\dfrac{(1-\lambda_{m/2}\mu_{m/2})C_{m/2}e^{\lambda_{m/2}\tau}}{1+\lambda_{m/2}\mu_j},\ldots,\dfrac{(1-\lambda_k\mu_k)C_{m+1-k}e^{-\lambda_k\tau}}{1-\lambda_k\mu_j},\ldots,\dfrac{(1-\lambda_1\mu_1)C_me^{-\lambda_1\tau}}{1-\lambda_1\mu_j}\right]} \times$$

$$(1-W)n^2I_b(\tau), \quad k = 1,\ldots,m/2 \tag{11}$$

where det[...] indicates the determinant. Now the following steps are carried out in the determinant of both the numerator and denominator of Eq. (11).

1) Factor out $(1-\lambda_1\mu_1)$ from the first and last columns, $(1-\lambda_2\mu_2)$ from the second column and the next to last column,..., and $(1-\lambda_{m/2}\mu_{m/2})$ from the $(m/2)$th and $(m/2+1)$th columns.

2) Factor out C_j from each column (i.e., factor out C_1 from the first column, C_2 from the second column,..., C_m from the last column).

3) Factor out $e^{\lambda_j\tau}$ from the first $(m/2)$ columns and $e^{-\lambda_j\tau}$ from the last $(m/2)$ columns (i.e., factor out $e^{\lambda_1\tau}$ from the first column,..., $e^{\lambda_{m/2}\tau}$ from the $(m/2)$th column, $e^{-\lambda_{m/2}\tau}$ from the $(m/2+1)$th column,..., and $e^{-\lambda_1\tau}$ from the last column).

Execution of the above steps results in the following expression

$$\frac{dv_{m+1-k}}{d\tau} = \frac{\prod_{i=1}^{m} C_i \prod_{i=1}^{m/2}(1-\lambda_i\mu_i)^2 \prod_{i=1}^{m/2} e^{-\lambda_i\tau} \prod_{i=1}^{m/2} e^{\lambda_i\tau}(1-W)n^2I_b(\tau)K_{m+1-k}}{\prod_{i=1}^{m} C_i \prod_{i=1}^{m/2}(1-\lambda_i\mu_i)^2 \prod_{i=1}^{m/2} e^{-\lambda_i\tau} \prod_{i=1}^{m/2} e^{\lambda_i\tau}C_{m+1-k}e^{-\lambda_k\tau}(1-\lambda_k\mu_k)},$$

$$k = 1,\ldots,m/2 \tag{12}$$

where

$$K_{m+1-k} = \frac{\det\left[\dfrac{1}{1+\lambda_1\mu_j},\ldots,\dfrac{1}{1+\lambda_{m/2}\mu_j},\ldots,\dfrac{1}{\mu_j},\ldots,\dfrac{1}{1-\lambda_1\mu_j}\right]}{\det\left[\dfrac{1}{1+\lambda_1\mu_j},\ldots,\dfrac{1}{1+\lambda_{m/2}\mu_j},\ldots,\dfrac{1}{1-\lambda_k\mu_j},\ldots,\dfrac{1}{1-\lambda_1\mu_j}\right]} \tag{13}$$

Now after like terms are cancelled in both the numerator and denominator, Eq. (12) yields

$$\frac{dv_{m+1-k}}{d\tau} = \frac{e^{\lambda_k\tau}(1-W)n^2I_b(\tau)}{C_{m+1-k}(1-\lambda_k\mu_k)}K_{m+1-k}, \quad k = 1,\ldots,m/2 \tag{14}$$

Note that K_{m+1-k} is not a function of τ but is rather some value which depends only on the eigenvalues and quadrature points. Proceeding as above, the expression for $dv_k/d\tau$ may be determined; therefore in general form it may be written that

$$\left.\begin{aligned} \frac{dv_k}{d\tau} &= K_k\frac{(1-W)n^2I_b(\tau)e^{-\lambda_k\tau}}{C_k(1-\lambda_k\mu_k)} \\ \frac{dv_{m+1-k}}{d\tau} &= K_{m+1-k}\frac{(1-W)n^2I_b(\tau)e^{\lambda_k\tau}}{C_{m+1-k}(1-\lambda_k\mu_k)} \end{aligned}\right\} k = 1,\ldots,m/2 \tag{15}$$

Upon solving the differential equations in Eq. (15), the solutions are given as

$$\left.\begin{aligned} v_k &= \frac{K_k(1-W)n^2}{C_k(1-\lambda_k\mu_k)}\int_{\tau_o}^{\tau} I_b(t)e^{-\lambda_k t}\,dt \\ v_{m+1-k} &= \frac{K_{m+1-k}(1-W)n^2}{C_{m+1-k}(1-\lambda_k\mu_k)}\int_{o}^{\tau} I_b(t)e^{\lambda_k t}\,dt \end{aligned}\right\} k = 1,\ldots,m/2 \tag{16}$$

where τ_o is the optical thickness and the limits of integration are chosen in accordance with Fig. 1.

Substituting the results from Eq. (16) into Eq. (8) reveals the particular solutions as

$$I_p(\tau,\mu_i) = (1-W)n^2\left[\int_o^\tau I_b(t)\sum_{j=1}^{m/2}\left\{\frac{K_{m+1-j}e^{-\lambda_j(\tau-t)}}{1-\lambda_j\mu_i}\right\}dt + \int_{\tau_o}^{\tau} I_b(t)\sum_{j=1}^{m/2}\left\{\frac{K_j}{1+\lambda_j\mu_i}e^{-\lambda_j(t-\tau)}\right\}dt\right], \quad i = 1,\ldots,m \tag{17}$$

Now to verify that Eq. (17) is a solution to Eq. (2), substitute Eq. (17) into Eq. (2). If Eq. (17) is truly a solution of Eq. (2), then Eq. (17) must satisfy Eq. (2). By using Leibnitz rule to evaluate the left-hand side of Eq. (2) and by using Eq. (7) to simplify the second term on the right-hand side of Eq. (2), it is found that in order for Eq. (17) to satisfy Eq. (2), K_j and K_{m+1-j} must satisfy the expression

$$\sum_{j=1}^{m/2}\{K_j/(1+\lambda_j\mu_i) + (K_{m+1-j})/(1-\lambda_j\mu_i)\} = 1/\mu_i \quad i = 1,\ldots,m \tag{18}$$

This is precisely the expression corresponding to Eq. (13), i.e., the K_j and K_{m+1-j} in Eq. (18) can readily be seen to satisfy an expression which has the form of Eq. (13). Thus it has been verified that Eq. (17) is a valid solution since K_j and K_{m+1-j} are required to satisfy the same expression both before and after substituting Eq. (17) into Eq. (2).

Determination of K_j and K_{m+1-j}

With the particular solutions given by Eq. (17) it would be especially convenient if the constants K_j and $K_{m+1-j}[j = 1,\ldots,m/2]$ could be determined in closed form. At this point in the derivation the constants must be determined through use of a library subroutine for solving systems of linear inhomogeneous algebraic equations; the system of equations to be solved is Eq. (18). It can readily be shown that

$$K_j = -K_{m+1-j}, \quad j = 1,\ldots,m/2 \tag{19}$$

if one writes the expression for K_j and K_{m+1-j} in the form of Eq. (13). Now if in the numerator of Eq. (13), the first and last rows are interchanged, the second and next to last rows are interchanged,..., and the $m/2$th and $(m/2+1)$th rows are interchanged, and then the first and last columns are interchanged, the second and next to last columns are interchanged,..., and the $m/2$th and $(m/2+1)$th columns are interchanged, the result given in Eq. (19) is readily observed. Inserting Eq. (19) into Eq. (18) yields

$$\sum_{j=1}^{m/2} -2K_j\lambda_j\mu_i^2/(1-\lambda_j^2\mu_i^2) = 1, \quad i = 1,\ldots,m/2 \tag{20}$$

where Eq. (20) supplies $m/2$ equations to be solved for the $m/2$ unknowns, K_j; the other $m/2$ unknowns are given by Eq. (19). Now if each ith equation in Eq. (20) is multiplied by a_i (quadrature weight) and then all $m/2$ equations are summed, the result is

$$\sum_{i=1}^{m/2} a_i \sum_{j=1}^{m/2} -2K_j\lambda_j\mu_i^2/(1-\lambda_j^2\mu_i^2) = \sum_{i=1}^{m/2} a_i = 1 \tag{21}$$

Now if the sums over i and j are interchanged then

$$-2\sum_{j=1}^{m/2}(K_j/\lambda_j)\left\{\sum_{i=1}^{m/2} a_i\lambda_j^2\mu_i^2/(1-\lambda_j^2\mu_i^2)\right\} = 1 \tag{22}$$

and using Eq. (5) to replace the sum over i, the result becomes

$$-2\sum_{j=1}^{m/2} K_j/\lambda_j = W/(1-W) \tag{23}$$

There are many possible choices of K_j which will satisfy Eq. (23); however, it is also likewise required that Eq. (20) be satisfied. In order to find the correct values of K_j needed in Eq. (20), a modified version of the method described in Ref. 4 entitled "elimination of the constants" will be used here. This consists of writing Eq. (18) in the form

$$S(\mu) = \sum_{j=1}^{m/2}\left\{K_j/(1+\lambda_j\mu) + (K_{m+1-j})/(1-\lambda_j\mu)\right\} - 1/\mu \tag{24}$$

which now replaces μ_i by μ thus allowing μ to be treated as a continuous variable instead of a fixed discrete value; also $S(\mu_\alpha) = 0\,[\alpha = 1,\ldots,m]$ in accordance with Eq. (18). If $S(\mu)$ is multiplied by

$$\mu\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha^2\mu^2) \quad \text{then} \quad S(\mu)\mu\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha^2\mu^2)$$

is a polynomial of degree m in μ which vanishes for $\mu = \mu_\alpha[\alpha = 1,\ldots,m]$. Hence

$$S(\mu)\mu\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha^2\mu^2)$$

cannot differ from the polynomial

$$P(\mu) = \prod_{\alpha=1}^{m/2}(\mu^2-\mu_\alpha^2) \tag{25}$$

by more than a constant factor A. Thus it may be written that

$$S(\mu)\mu\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha^2\mu^2) = A\prod_{\alpha=1}^{m/2}(\mu^2-\mu_\alpha^2) \tag{26}$$

or

$$S(\mu) = A\prod_{\alpha=1}^{m/2}(\mu^2-\mu_\alpha^2)\Big/\mu\prod_{\alpha=1}^{m/2}(1-\lambda_\alpha^2\mu^2) \tag{27}$$

Explicit expressions for the constants K_j and $K_{m+1-j}[j = 1,\ldots,m/2]$ can now be found in the following manner. From Eq. (24) it is apparent that

$$K_j = \lim_{\mu\to -1/\lambda_j}(1+\lambda_j\mu)S(\mu) = \frac{-A\lambda_j P(-1/\lambda_j)}{2R_j(-1/\lambda_j)} \tag{28}$$

and

$$K_{m+1-j} = \lim_{\mu\to 1/\lambda_j}(1-\lambda_j\mu)S(\mu) = \frac{A\lambda_j P(1/\lambda_j)}{2R_j(1/\lambda_j)} \tag{29}$$

where

$$R_j(\mu) = \prod_{\substack{\alpha=1\\ \alpha\neq j}}^{m/2}(1-\lambda_\alpha^2\mu^2) \tag{30}$$

and $P(\mu)$ is given in Eq. (25). It is also apparent that $R_j(-1/\lambda_j) = R_j(1/\lambda_j)$ and $P(1/\lambda_j) = P(-1/\lambda_j)$; thus it is seen from Eqs. (28) and (29) that $K_j = -K_{m+1-j}$ as was indicated in Eq. (19). Hence, by two independent derivations it has been shown that $K_j = -K_{m+1-j}$. From Eqs. (28) and (29) it now becomes obvious that if the constant A can be determined then the values of K_j and K_{m+1-j} will also be known. The value of A is readily found by substituting Eq. (28) into Eq. (23); the result is

$$A = \frac{W}{1-W}\left\{\frac{1}{\sum_{\alpha=1}^{m/2} P(1/\lambda_\alpha)/R_\alpha(1/\lambda_\alpha)}\right\} \tag{31}$$

The constant A can also be determined by an alternate procedure as shown in Appendix A.

In summary the solution for the constants K_j and K_{m+1-j} $[j = 1,\ldots,m/2]$ of Eq. (18) are given by

$$K_{m+1-j} = -K_j = \frac{A\lambda_j P(1/\lambda_j)}{R_j(1/\lambda_j)}, \quad j = 1,\ldots,m/2 \tag{32}$$

where A is given in Eq. (31), $P(1/\lambda_j)$ is given from Eq. (25), and $R_j(1/\lambda_j)$ is given from Eq. (30). These values of K_j and K_{m+1-j} given in Eq. (32) are the closed-form solutions for the constants needed to complete the particular solutions given by Eq. (17). The values of the constants K_j and K_{m+1-j} have been computed by three independent methods: 1) solution of Eq. (18) directly by a computer library routine (Cholesky method[7]), 2) solution of Eq. (20) directly by a computer library routine,[7] and 3) computation directly from Eq. (32). The agreement given by all three techniques was excellent. Also, the values of K_j and K_{m+1-j} determined by these three methods were inserted back into Eq. (18) to be sure that Eq. (18) was satisfied for all $i = 1,\ldots,m$. The values of K_j and K_{m+1-j} were computed for $W = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99, and 0.9999. Also computation was done for various orders of Gaussian quadrature. Shown in Tables 1 and 2 are the values of $K_{m+1-j} = -K_j$ for a variety of W and for both 32 and 48 point single Gaussian quadratures. It should be noted that the solution given by Eq. (32) can be used for any specified order of Gaussian quadrature and is valid for either the single (even order) or double Gaussian quadrature.

Table 1 Values of $K_{m+1-j} = -K_j$ for various values of W^a

j	W = 0.10	W = 0.50	W = 0.90	W = 0.99	W = 0.9999
1	0. 0006985	0. 1910948	2. 1602636	8. 4704731	86. 5824869
2	0. 0013519	0. 0035725	0. 0008930	0. 0007475	0. 0007341
3	0. 0019317	0. 0097671	0. 0031175	0. 0026161	0. 0025697
4	0. 0024916	0. 0150445	0. 0065078	0. 0055274	0. 0054348
5	0. 0030604	0. 0191599	0. 0110131	0. 0095360	0. 0093920
6	0. 0036632	0. 0226476	0. 0166152	0. 0147434	0. 0145530
7	0. 0043278	0. 0259878	0. 0233571	0. 0213113	0. 0210916
8	0. 0050898	0. 0295631	0. 0313880	0. 0294972	0. 0292779
9	0. 0060004	0. 0337432	0. 0410292	0. 0397191	0. 0395435
10	0. 0071388	0. 0389885	0. 0528898	0. 0526820	0. 0526077
11	0. 0086392	0. 0460057	0. 0681027	0. 0696392	0. 0697410
12	0. 0107526	0. 0560690	0. 0888754	0. 0930029	0. 0933817
13	0. 0140136	0. 0718627	0. 1199896	0. 1279997	0. 1288074
14	0. 0198091	0. 1003272	0. 1739119	0. 1882968	0. 1898176
15	0. 0332202	0. 1668694	0. 2963650	0. 3241338	0. 3271475
16	0. 0999625	0. 5000686	0. 8987914	0. 9880373	0. 9978400

^{a}32-point single Gaussian quadrature was used in Eq. (32) [m = 32]

Table 2 Values of $K_{m+1-j} = -K_j$ for various values of W.^{a}

j	W = 0.10	W = 0.50	W = 0.90	W = 0.99	W = 0.9999
1	0. 0003519	0. 1910948	2. 1602636	8. 4704727	86. 5836987
2	0. 0006721	0. 0010956	0. 0002907	0. 0002455	0. 0002414
3	0. 0009503	0. 0034512	0. 0009876	0. 0008315	0. 0008171
4	0. 0012105	0. 0060846	0. 0020300	0. 0017130	0. 0016836
5	0. 0014626	0. 0084907	0. 0034043	0. 0028916	0. 0028433
6	0. 0017131	0. 0105548	0. 0051012	0. 0043764	0. 0043070
7	0. 0019672	0. 0123339	0. 0071141	0. 0061822	0. 0060911
8	0. 0022299	0. 0139249	0. 0094409	0. 0083293	0. 0082178
9	0. 0025062	0. 0154205	0. 0120866	0. 0108452	0. 0107168
10	0. 0028021	0. 0169005	0. 0150669	0. 0137669	0. 0136275
11	0. 0031246	0. 0184358	0. 0184123	0. 0171450	0. 0170028
12	0. 0034824	0. 0200953	0. 0221732	0. 0210486	0. 0209143
13	0. 0038869	0. 0219529	0. 0264282	0. 0255738	0. 0254607
14	0. 0043534	0. 0240972	0. 0312959	0. 0308572	0. 0307812
15	0. 0049035	0. 0266436	0. 0369534	0. 0370964	0. 0370765
16	0. 0055683	0. 0297541	0. 0436696	0. 0445866	0. 0446454
17	0. 0063956	0. 0336704	0. 0518634	0. 0537856	0. 0539506
18	0. 0074618	0. 0387781	0. 0622173	0. 0654387	0. 0657447
19	0. 0088990	0. 0457390	0. 0759142	0. 0808399	0. 0813344
20	0. 0109563	0. 0557989	0. 0951919	0. 1024420	0. 1031959
21	0. 0141666	0. 0716242	0. 1248465	0. 1355108	0. 1366467
22	0. 0199156	0. 1001430	0. 1773476	0. 1937520	0. 1955282
23	0. 0332830	0. 1667537	0. 2984119	0. 3274428	0. 3306185
24	0. 0999833	0. 5000292	0. 8994712	0. 9891462	0. 9990044

^{a}48-point single Gaussian quadrature was used in Eq. (32) [m = 48]

Discussion and Conclusions

The particular solutions to the radiative transport equation for one-dimensional axisymmetric transport with isotropic scattering have been derived and are given by Eq. (17) with the constants K_j and K_{m+1-j} given by Eq. (32). The eigenvalues to be used in Eqs. (17) and (32) are easily determined from either Eqs. (5, 6, or 7) as shown in Ref. 5. Combining the particular and homogeneous solutions, the general solution may be expressed as

$$I(\tau,\mu_i)=\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\left\{\frac{C_je^{\lambda_j\tau}}{1+\lambda_j\mu_i}+\frac{C_{m+1-j}e^{-\lambda_j\tau}}{1-\lambda_j\mu_i}\right\}+(1-W)n^2\left[\int_o^\tau I_b(t)\sum_{j=1}^{m/2}\left\{\frac{K_{m+1-j}e^{-\lambda_j(\tau-t)}}{1-\lambda_j\mu_i}\right\}dt+\int_{\tau_o}^{\tau} I_b(t)\sum_{j=1}^{m/2}\left\{\frac{K_je^{-\lambda_j(t-\tau)}}{1+\lambda_j\mu_i}\right\}dt\right],\quad i=1,\ldots,m \tag{33}$$

where C_j and $C_{m+1-j}[j=1,\ldots,m/2]$ are the constants of integration to be determined through use of the specific boundary conditions applied to the given problem. These integration constants can be readily obtained by standard computer library routines such as the Gauss-Jordon method. The particular solutions, Eq. (17), and the general solution, Eq. (33), are valid for a medium having one-dimensional axisymmetric, radiation transport with absorption, emission, and isotropic scattering. Also, the temperature profile across the medium may be arbitrary.

Finally, Eq. (33) can be used to derive an expression for the radiative flux. Consistent with the axisymmetric assumption, the radiative flux at optical depth τ in the medium is defined as

$$q_R(\tau)=2\pi\int_{-1}^{1}I(\tau,\mu)\mu d\mu \tag{34}$$

or in terms of the Gaussian quadrature

$$q_R(\tau)=2\pi\sum_{i=1}^{m}I(\tau,\mu_i)\mu_i a_i \tag{35}$$

Substitution of Eq. (33) into Eq. (35) and rearranging yields

$$q_R(\tau)=2\pi\sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\left\{C_je^{\lambda_j\tau}\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}+C_{p+1-j}e^{-\lambda_j\tau}\sum_{i=1}^{m}\frac{a_i\mu_i}{1-\lambda_j\mu_i}\right\}+2\pi(1-W)n^2\left\{\int_o^\tau I_b(t)\sum_{j=1}^{m/2}K_{m+1-j}e^{-\lambda_j(\tau-t)}\sum_{i=1}^{m}\frac{a_i\mu_i}{1-\lambda_j\mu_i}dt+\int_{\tau_o}^{\tau}I_b(t)\sum_{j=1}^{m/2}K_je^{-\lambda_j(t-\tau)}\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}dt\right\} \tag{36}$$

The summations over index i can be evaluated as follows:

$$\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}=\sum_{i=1}^{m/2}\frac{a_i\mu_i}{1+\lambda_j\mu_i}+\sum_{i=m/2+1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i} \tag{37}$$

or

$$\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}=\sum_{i=1}^{m/2}\frac{a_i\mu_i}{1+\lambda_j\mu_i}+\sum_{i=1}^{m/2}\frac{a_{m+1-i}\mu_{m+1-i}}{1+\lambda_j\mu_{m+1-i}} \tag{38}$$

Recalling that $\mu_k=-\mu_{m+1-k}$, $a_k=a_{m+1-k}[k=1,\ldots,m/2]$, and then applying these relationships to the second term on the right-hand side of Eq. (38) gives, after simplifying

$$\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}=\frac{-2}{\lambda_j}\sum_{i=1}^{m/2}\frac{a_i\mu_i^2\lambda_j^2}{1-\lambda_j^2\mu_i^2} \tag{39}$$

Now using Eq. (5) to replace the summation in Eq. (39), this becomes

$$\sum_{i=1}^{m}\frac{a_i\mu_i}{1+\lambda_j\mu_i}=\frac{-2}{\lambda_j}\frac{(1-W)}{W} \tag{40}$$

and similarly

$$\sum_{i=1}^{m}\frac{a_i\mu_i}{1-\lambda_j\mu_i}=\frac{2}{\lambda_j}\frac{(1-W)}{W} \tag{41}$$

Substituting Eqs. (40) and (41) into Eq. (36) to evaluate the summations over index i, the expression for the radiative flux becomes

$$q_R(\tau)=-4\pi\frac{(1-W)}{W}\sum_{j=1}^{m/2}\frac{(1-\lambda_j\mu_j)}{\lambda_j}\left\{C_je^{\lambda_j\tau}-C_{m+1-j}e^{-\lambda_j\tau}\right\}+\frac{4\pi(1-W)^2n^2}{W}\left\{\int_o^\tau I_b(t)\sum_{j=1}^{m/2}\frac{K_{m+1-j}}{\lambda_j}e^{-\lambda_j(\tau-t)}dt-\int_{\tau_o}^{\tau}I_b(t)\sum_{j=1}^{m/2}\frac{K_j}{\lambda_j}e^{-\lambda_j(t-\tau)}dt\right\} \tag{42}$$

Equation (42) describes the radiative flux at optical depth τ in terms of the governing parameters, the integration constants C_j and C_{m+1-j}, and also in terms of the particular solution constants K_j and K_{m+1-j}.

In the derivation of the above equations isotropic scattering was assumed because in many practical problems the phase function is unknown.[8,9] Therefore, the assumption of isotropic scattering is generally employed to obtain a baseline calculation. The method used in Ref. 1, which is also applicable to isotropic scattering, requires a relatively large amount of numerical computation. The technique of Ref. 1 requires the computation of the characteristic function, which necessitates determining the coefficients of the characteristic function; it also involves evaluation of the first derivative of the characteristic function. The solution presented in this paper yields the particular solution constants in a form which is simple and fast for computation. There is no need to determine the coefficients of a characteristic function or its derivative. Also, the method presented here has the advantage of only requiring $m/2$ values of $K_j[j=1,\ldots,m/2]$, since the other $m/2$ values are given immediately by Eq. (19); in Ref. 1 it was not shown that this simplifying relationship occurs between the idempotents. Thus, via Eq. (19), the method presented above reduces the effort in determining the particular solution constants at least in half as opposed to Ref. 1. In addition, Ref. 1 does not have a simple expression for the radiative flux [(such as Eq. (42)]. Appendix B presents the results of an illustrative example where two linear temperature profiles have been utilized in Eq. (42).

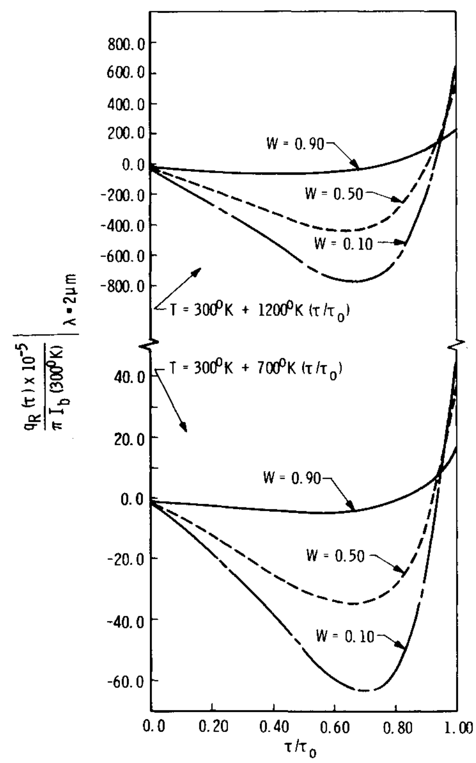

Fig. 2 Dimensionless monochromatic radiative flux for an absorbing, emitting, and scattering coating ($n = 1.2$, $\tau_o = 1.0$) on a gold substrate ($\bar{n} = 1.31$–$i10.70$).

Appendix A: Constant A

Rather than using Eq. (31) for determining the constant A, a simple and more direct method can be utilized. From Eqs. (24) and (27) it is apparent that

$$\lim_{\mu \to 0} \mu S(\mu) = A(-1)^{m/2} \prod_{\alpha=1}^{m/2} \mu_\alpha^2 = -1 \tag{A1}$$

or

$$A = (-1)^{1-m/2} \Big/ \prod_{\alpha=1}^{m/2} \mu_\alpha^2 \tag{A2}$$

The numerical results obtained by employing Eq. (31) or Eq. (A2) for determining A are identical; however, as is obvious, Eq. (A2) is much simpler and faster to use than Eq. (31). Also, Eq. (A2) shows A to be only a function of the quadrature points.

Appendix B: Illustrative Example

In order to show a simple application of the method discussed in this paper, a sample problem has been included to show some illustrative results. The sample problem consists of an absorbing, emitting and scattering coating bounded by a gold substrate (refractive index[10] $n = 1.31 - i10.70$) and bounded at the top ($\tau = \tau_o$) by air. The substrate is considered to be held at a constant temperature of 300 K with the top interface taken to be either 1000 K or 1500 K. A linear temperature profile is assumed to exist between the substrate and top interface for these two cases; the optical thickness of the medium is taken to be unity ($\tau_o = 1.0$). The substrate and top interface are assumed to be specular surfaces which reflect and transmit radiation in accordance with Fresnel's equations and Snell's law; the coating refractive index is 1.2. Shown in Fig. 2 are the nondimensional radiative flux [(Eq. (42)] profiles as a function of local optical thickness with albedo taken as a parameter for two linear temperature profiles. The radiative flux at any local optical thickness is nondimensionalized by the flux of a 300 K blackbody in vacuum; all results are monochromatic at a wavelength[10] of $2\mu m$. A 24-point single Gaussian quadrature was employed to obtain the results shown in Fig. 2.

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